3 Sobolev Spaces

Exercise 3.1. Let $I = (0, 1)$. Given $u \in W^{1,1}(I)$, (by definition) there exists $v \in L^1(I)$ such that

$$u(x) = u(y) + \int_y^x v(t)dt$$

for all $x, y \in I$. Show that v coincides with the distributional derivative of u.

Exercise 3.2. Let $I = (0, 1)$ and (u, u') and (v, v') be two elements of $W^{1,1}(I) \subset L^1(I) \times L^1(I)$ such that $u = v$ in $L^1(I)$, then $u' = v'$ in $L^1(I)$.

Exercise 3.3. Let $I = (0, 1)$.
— Show that, in general, step functions are not in $W^{1,1}(I)$.
— Compute for which values of $\alpha \in \mathbb{R}$ the function x^α is in $W^{1,1}(I)$.

Exercise 3.4. Let $I = [0, 1] \subset \mathbb{R}$. Show that the set

$$W_0^{1,1}(I) = \left\{u \in W^{1,1}(I) : u(0) = u(1) = 0\right\}$$

is a closed subspace of $W^{1,1}(I)$. In particular, $W_0^{1,1}(I)$ is a Banach space. Moreover, prove that for every $u \in W_0^{1,1}(I)$ there are $a, b > 0$ such that

$$a\|u'\|_{L^1(I)} \leq \|u\|_{W^{1,1}(I)} \leq b\|u'\|_{L^1(I)}.$$

Exercise 3.5. Let $1 \leq p < \infty$ and let $I = (0, 1)$. Show that $W^{1,p}(I) \subseteq C^{0,1-1/p}(I)$, where $C^{0,1-1/p}(I)$ is the usual space of Hölder continuous functions. Moreover, show that the opposite inclusion fails.